Some continuous analogs of expansion in Jacobi polynomials and vector valued hypergeometric orthogonal bases

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Abstract. We write spectral decomposition of the hypergeometric differential operator on the contour $\mathrm{Re}\, z = 1/2$ (multiplicity of spectrum is 2). As a result, we obtain an integral transform that differs from the Jacobi (or Olevsky) transform. We also try to do a step towards vector-valued special functions and construct a ${}_3F_2$-orthogonal basis in the space of functions having values in 2-dimensional space. This basis is lying in an analytic continuation of continuous dual Hahn polynomials with respect to number n of a polynomial.

In Addendum, we discuss $\mathbb{C}^2$-value analogs of the Meixner–Pollachek orthogonal system and also perturbations of Laguerre, Meixner, and Jacobi polynomials.

§1. Representation-theoretical motivation and formulation of results.

1.1. Continuous analogs of expansion in Jacobi polynomials. The present work is an counterpart on the level of special functions of Molchanov's paper [17] on tensor products of unitary representations of the group $\mathrm{SL}_2(\mathbb{R})$.

In the classical analysis, there are well-known the expansion in Jacobi polynomials and also its continuous analogue. The latter is the index hypergeometric transform of H.Weyl, [33] (it is also called by Olevsky transform [23], Jacobi transform, generalized Fourier transform, see Koornwinder's survey [13], see also [8], [12], [20]). These classical constructions have a transparent and important representation theoretic interpretation (see [32]). Spherical functions of the projective spaces

$$\mathrm{O}(n+1)/\mathrm{O}(n)\times\mathrm{O}(1), \qquad \mathrm{U}(n+1)/\mathrm{U}(n)\times\mathrm{U}(1), \qquad \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\times\mathrm{Sp}(1) \quad (1.1)$$

over $\mathbb{R}$, $\mathbb{C}$ and the quaternion field $\mathbb{H}$ are the Jacobi polynomials $P_m^{\alpha,\beta}$ for some special values of the parameters α, β. The theorem about decomposition of L^2 on these spaces into a direct sum of irreducible representations is a corollary of the theorem on expansion of a function into a series in Jacobi polynomials.

For hyperbolic spaces

$$\mathrm{O}(n,1)/\mathrm{O}(n)\times\mathrm{O}(1), \qquad \mathrm{U}(n,1)/\mathrm{U}(n)\times\mathrm{U}(1), \qquad \mathrm{Sp}(n,1)/\mathrm{Sp}(n)\times\mathrm{Sp}(1) \quad (1.2)$$

the analogy of an expansion in the Jacobi polynomials is the *index hypergeometric transform* (see [23], [13])

$$g(s) = \frac{1}{\Gamma(b+c)}\int_0^\infty f(x)\ {}_2F_1(b+is, b-is; b+c; -x)x^{b+c-1}(1+x)^{b-c}dx. \quad (1.3)$$

The problem of decomposition of L^2 on these spaces is reduced to the inversion formula

$$f(x) = \frac{1}{\pi\Gamma(b+c)} \int_0^\infty g(s)\ {}_2F_1(b+is, b-is; b+c; -x)\Bigl|\frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)}\Bigr|^2 ds$$

for this integral transform.

Nevertheless, these two classical constructions (i.e., the expansion in the Jacobi polynomials and the index hypergeometric transform are not sufficient in the analysis on pseudo-Riemannian symmetric spaces of rank 1 (this class of spaces includes, in particular, other real forms of the spaces (1.1)–(1.2)). This force to think that there exists some another analog (or analogs) of the index hypergeometric transform (1.3). We construct one such transform[1] [2].

1.2. The problem on vector-valued bases. The Askey–Wilson hierarchy of hypergeometric orthogonal polynomials is well known, see for instance, [11], or [1], Chapter 6. Almost all these polynomials (probably all) appear in the representation theory of the group $\mathrm{SL}_2(\mathbb{R})$ (see, for instance, [32], [28]–[29]).

Consider the tensor product of a unitary highest weight representation of $\mathrm{SL}_2(\mathbb{R})$ and a lowest weight representation. In each factor, there is a canonical orthogonal basis consisting of $\mathrm{SO}(2)$-eigenfunctions. Hence there exists a canonical basis in the tensor product. This basis consists of continuous dual Hahn polynomials ([34]). Recall (see [1], 6.10, [11]) that they are the polynomials $p_n(s^2)$ orthogonal with respect to the measure

$$\Bigl|\frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)}\Bigr|^2 ds$$

on the line $s \in \mathbb{R}$; the explicit formula is

$$p_n(s^2) = (a+b)_n (a+c)_n\ {}_3F_2\left[\begin{matrix}-n, a+is, a-is \\ a+b, a+c\end{matrix}; 1\right]. \tag{1.4}$$

But the same problem has sense for each pair of unitary representations of $\mathrm{SL}_2(\mathbb{R})$ or its universal covering. These tensor products (generally) have multiplicity 2 (see [25], [17]). Hence, we must obtain some orthogonal bases consisting of $\mathbb{C}^2$-valued functions.

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[1] Molchanov [17], [18] [19] uses another method. Below we introduce the hypergeometric differential operator (2.1). It arises if we restrict the Laplace operator in tensor product of two unitary representations of the universal covering group $\mathrm{SL}_2(\mathbb{R})^\sim$ of $\mathrm{SL}_2(\mathbb{R})$ to eigenspaces of the rotation group $\mathrm{SO}(2)$. Molchanov [17] (he consider only the group $\mathrm{PSL}_2(\mathbb{R})$) restricts the Laplace operator to functions that are invariant with respect to the group of diagonal matrices. As a result, he obtains the Legendre differential operator on some contour containing singular points of the Legendre equation.

[2] It is interesting to understand, is it possible to give another proof of the Molchanov's Plancherel formula [18] for rank 1 symmetric spaces G/H using the classical index hypergeometric transform (in the extended variant [5], XIII or [8]) and the double index hypergeometric transform constructed below in 1.4. Precisely, let K be a maximal compact subgroup in G , let V be an irreducible K-module. Consider the action of the Laplace operator on $\mathrm{Hom}_K(V, L^2(G/H))$. Is it correct, that in all the cases we will obtain the hypergeometric operator (2.3) on the segment $[0,1]$ (the classical case) or on the contour $\mathrm{Re}\, z = 1/2$ (our case)?

1.3. Notations. We use the standard notations for the Pochhammer symbol

$$(a)_n = a(a+1)\dots(a+n-1)$$

and for hypergeometric functions

$$\begin{aligned} {}_2F_1[a,b;c;z] &:= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n;\\ {}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};z\right] &:= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!} z^n. \end{aligned}$$

1.4. Double index hypergeometric transform. Fix $0 \leqslant \alpha \leqslant 1/2$, $\beta \in \mathbb{R}$. Assume that $\alpha + i\beta \neq 0$.

Consider the space of $\mathbb{C}^2$-valued functions of the half-line $s > 0$. An element of this space can be considered as a pair of scalar-valued functions $(\varphi_1(s), \varphi_2(s))$. Let us introduce the scalar product in this space by the formula

$$\begin{aligned} \big\langle(\varphi_1,\varphi_2),(\psi_1,\psi_2)\big\rangle = \frac{1}{2\pi}\int_0^\infty \Big[r_{11}(s)\varphi_1(s)\overline{\psi_1(s)} + r_{12}(s)\varphi_1(s)\overline{\psi_2(s)} + \\ + r_{21}(s)\varphi_2(s)\overline{\psi_1(s)} + r_{22}(s)\varphi_2(s)\overline{\psi_2(s)}\Big] \frac{ds}{|\Gamma(2is)|^2}, \end{aligned}$$

where $r_{ij}(s)$ are given by

$$\begin{aligned} R(s) = \begin{pmatrix} r_{11}(s) & r_{12}(s)\\ r_{21}(s) & r_{22}(s)\end{pmatrix} := \\ := \begin{pmatrix} \Gamma(\frac12-\alpha-is)\Gamma(\frac12-\alpha+is) & \Gamma(\frac12-i\beta-is)\Gamma(\frac12-i\beta+is)\\ \Gamma(\frac12+i\beta-is)\Gamma(\frac12+i\beta+is) & \Gamma(\frac12+\alpha-is)\Gamma(\frac12+\alpha+is)\end{pmatrix}. \end{aligned} \tag{1.5}$$

It is convenient to write this scalar product in the vector form

$$\big\langle(\varphi_1,\varphi_2),(\psi_1,\psi_2)\big\rangle = \frac{1}{2\pi}\int_0^\infty \begin{pmatrix}\varphi_1(s) & \varphi_2(s)\end{pmatrix} R(s) \begin{pmatrix}\overline{\psi_1(s)}\\ \overline{\psi_2(s)}\end{pmatrix} \frac{ds}{|\Gamma(2is)|^2}.$$

Thus we obtain the Hilbert space of $\mathbb{C}^2$-valued functions. We denote it by $H_{\alpha,\beta}$.

Let $x \in \mathbb{R}$, $s \geqslant 0$. Consider two functions $Q_1(\alpha,\beta;x,s)$, $Q_2(\alpha,\beta;x,s)$ given by the formulae[3]

$$\begin{aligned} Q_1(\alpha,\beta;x,s) = \frac{1}{\Gamma(\alpha+i\beta)}(\tfrac12+ix)^{-(\alpha+i\beta)/2}(\tfrac12-ix)^{(\alpha+i\beta)/2-is-1/2}\times \\ \times\ {}_2F_1\left[\begin{matrix}1/2-\alpha+is, 1/2-i\beta+is\\ 1-\alpha-i\beta\end{matrix}; \frac{ix+1/2}{ix-1/2}\right]; \end{aligned} \tag{1.6}$$

[3] Q_1, Q_2 are almost L^2-eigenfunctions of the differential operator D defined below (2.1).

$Q_2(\alpha,\beta;x,s) = Q_1(-\alpha,-\beta;x,s)$.

For a function $f \in L^2(\mathbb{R})$, we define the pair of functions $\varphi_1(s)$, $\varphi_2(s)$ by

$$\varphi_j(s) := \int_{-\infty}^{\infty} f(x)\overline{Q_j(\alpha,\beta;x,s)}\,dx, \qquad j = 1,2. \tag{1.7}$$

THEOREM 1.1. *Let* $0 \leqslant \alpha \leqslant 1/2$, $\alpha + i\beta \neq 0$.

a) *The operator* $f \mapsto (\varphi_1, \varphi_2)$ *is a unitary bijective operator from* $L^2(\mathbb{R})$ *to* $H_{\alpha,\beta}$.

b) *The inversion formula is*

$$f(x) := \frac{1}{2\pi}\int_0^{\infty} \begin{pmatrix} Q_1(\alpha,\beta;x,s) & Q_2(\alpha,\beta;x,s)\end{pmatrix} R(s) \begin{pmatrix}\varphi_1(s)\\ \varphi_2(s)\end{pmatrix} \frac{ds}{|\Gamma(2is)|^2}. \tag{1.8}$$

REMARK. Let $A(s)$ be a 2×2-matrix valued function on $\mathbb{R}$. Any calibration

$$(Q_1^\circ(s), Q_2^\circ(s)) := (Q_1(s), Q_2(s))A(s)^{-1}, \qquad R^\circ(s) := A(s)^* R(s) A(s)$$

gives another form of our integral transformation. It is interesting to find matrices $A(s)$, for which the matrix $R^\circ(s)$ and the vector $(Q_1^\circ(s), Q_2^\circ(s))$ remain relatively simple.

1.5. Generalization for $\alpha > 1/2$. Now, let $\alpha > 1/2$, $\beta \in \mathbb{R}$, and $\alpha + i\beta \neq 0, 1, 2, \ldots$. Denote by n the integral part of $\alpha - 1/2$. Consider the finite dimensional linear space $W_{\alpha,\beta}$, consisting of vectors $(c_0, c_1, \ldots, c_n)$. The scalar product in $W_{\alpha,\beta}$ is given by

$$\langle c, c'\rangle = \frac{1}{2\pi}\sum_{k=0}^{n} \frac{2\alpha - 2k - 1}{\Gamma(2\alpha - k)k!} c_k \overline{c}_k{}'.$$

Next, define the functions[4]

$$R(\alpha,\beta;x;k) :=$$

$$:= \frac{1}{\Gamma(\alpha+i\beta)}(\tfrac12 + ix)^{-(\alpha+i\beta)/2}(\tfrac12 - ix)^{-(\alpha-i\beta)/2}\ {}_2F_1\left[\begin{matrix}-k, k-2\alpha+1\\ 1-\alpha-i\beta\end{matrix}; \tfrac12 + ix\right]. \tag{1.9}$$

Now, consider the linear operator

$$J_{\alpha,\beta} : L^2(\mathbb{R}) \to H_{\alpha,\beta} \oplus W_{\alpha,\beta}, \tag{1.10}$$

given by

$$f \mapsto (\varphi_1, \varphi_2, \theta),$$

where φ_1, φ_2 is defined by (1.7) as above, and the coordinates of the vector $\theta \in W_{\alpha,\beta}$ have the form

$$\theta_k = \int_{-\infty}^{\infty} f(x)\overline{R(\alpha,\beta;x;k)}\,dx.$$

[4]These functions are L^2-eigenfunctions of the differential operator D defined below

THEOREM 1.2. *The operator $J_{\alpha,\beta}$ is a unitary bijective operator.*

1.6. Romanovski polynomials. The Romanovski polynomials [27] are the polynomials on $\mathbb{R}$ orthogonal with respect to the weight

$$(\tfrac{1}{2}+ix)^{-(\alpha+i\beta)}(\tfrac{1}{2}-ix)^{-(\alpha-i\beta)}\,dx.$$

This weight decreases as $x^{-2\alpha}$, and hence it is possible to orthogonalize only finite number of power functions 1, x, x^2, ... Romanovski polynomials are defined by the formula

$${}_2F_1[-k,k-2\alpha+1;1-\alpha-i\beta;\tfrac{1}{2}+ix],$$

i.e., they coincide with (1.9) up to an elementary factor.

Recall that the Jacobi polynomials are given by

$$P_n^{\gamma,\delta}(x)=\mathrm{const}\cdot\ {}_2F_1[-n,n+\gamma+\delta,\delta+1;(1+x)/2]$$

We observe that the Romanovski polynomials are analytic continuations of the Jacobi polynomials with respect to the superscripts.

Many orthogonal systems of this kind are known, see [2], [14]–[16], [3], [21], see also [24], [3], [4] for some applications. Since these systems are eigenfunctions of some second order differential or difference operators, we obtain a good collection of spectral problems (and hence this must give a collection of new integral transforms).[5] Recently, one of the most complicated problems of this kind (related to one of families of hypergeometric ${}_4F_3$-polynomials from [21]) was solved by Groenevelt [9].

1.7. Vector-valued bases. Fix parameters $0<\alpha<1/2$, $\beta,p,q\in\mathbb{R}$. We consider the Hilbert space $Y(\alpha,\beta;q)$, consisting of $\mathbb{C}^2$-valued functions on half-line $s\geqslant 0$ with the scalar product

$$\begin{aligned}\langle(\varphi_1,\varphi_2),(\psi_1,\psi_2)\rangle&=\\ =\frac{1}{2}\int_0^\infty&\begin{pmatrix}\varphi_1(s)&\varphi_2(s)\end{pmatrix}R(s)\begin{pmatrix}\overline{\psi_1(s)}\\ \overline{\psi_2(s)}\end{pmatrix}\Bigl|\frac{\Gamma(1+iq+is)\Gamma(1-iq+is)}{\Gamma(2is)}\Bigr|^2ds,\end{aligned}\tag{1.11}$$

where the matrix R is the same as above (1.6).

Next, define functions $\Xi_n^{(1)}(\alpha,\beta;p,q;s)$, $\Xi_n^{(2)}(\alpha,\beta;p,q;s)$ in the variable s,

[5]It seems,that now only difference operators remain interesting. There are 3 types of difference operators related to

a) Shift operator on the lattice $\mathbb{Z}$: $Tf(n)=f(n+1)$.

b) Shift operator on the line $\mathbb{R}$: $Tf(x)=f(x+1)$.

c) Shift operator on the line $\mathbb{R}$ in the imaginary direction $Tf(x)=f(x+i)$.

It seems, that for the case b), we have infinite multiplicities. In any case, variants a), c) are interesting for our purposes.

given by the formula

$$\Xi_n^{(1)}(\alpha,\beta;p,q;s)=\frac{\cos(p-\alpha-iq+i\beta)\pi/2\,\Gamma(1-\alpha+i\beta)}{\Gamma(\alpha-i\beta)\Gamma(1+iq+i\beta)\Gamma(1+iq-\alpha)}\times$$
$$\times\ {}_3F_2\left[\begin{matrix}(1-\alpha+i\beta-p+iq)/2-n,\,1/2+iq+is,1/2+iq-is\\ 1+iq+i\beta,1+iq-\alpha\end{matrix};1\right];\quad(1.12)$$

$$\Xi_n^{(2)}(\alpha,\beta;p,q;s)=\Xi_n^{(1)}(-\alpha,-\beta;p,q;x).$$

Remark. The hypergeometric series ${}_3F_2(a_1,a_2,a_2;b_1,b_2;1)$ is absolutely convergent for $\sum a_i<\sum b_j$ and admits an analytic continuation as a meromorphic (sigle-valued) function to arbitrary values of the parameters a_1, a_2, a_2, b_1, b_2. We understand (1.12) as a value of this meromorphic function. Below (3.4), we represent the function $\Xi^{(1)}$ as a hypergeometric series that converges for all interesting for us values of the parameters.

Theorem 1.3. *The system $(\Xi_n^{(1)},\Xi_n^{(2)})$, where n ranges in $\mathbb{Z}$, is an orthonormal basis of the Hilbert space $Y(\alpha,\beta;q)$.*

Remark. Emphasis, that the expression (1.12) has the structure

$$\mathrm{const}\cdot\ {}_3F_2\left[\begin{matrix}-n+h,a+is,a-is\\ a+b,a+c\end{matrix};1\right]\qquad(1.13)$$

with

$$\operatorname{Re}a=1/2,\qquad\operatorname{Re}b=1/2,\qquad c\in\mathbb{R}.$$

Also, in comparison with formula (1.4) for the Hahn polynomials, the parameter n is shifted by some complex value h.

Remark. Similar perturbations exist for other classical hypergeometric systems (Laguerre, Jacobi, Meixner–Pollachek, Meixner), they are discussed in [22]).[6]

1.8. Further structure of the paper. Theorems 1.1–1.2 are proved in §2, Theorem 1.3 is obtained in §3.

In Addendum we discuss perturbations of Laguerre, Meixner, Meixner–Pollachek, and Jacobi systems.

§2. Spectral decomposition of the hypergeometric differential operator on the contour $\operatorname{Re}z=1/2$.

A proof of Theorems 1.1–1.2 proposed below is direct but very tedious. We directly apply the Weyl–Titchmarsh–Kodaira theorem for an appropriate differential operator. This theorem with lot of examples is analyzed in Chapter XIII of Dunford–Schwartz's book [5], see also the Titchmarsh's book [31].

[6] An attempt to find some ways for vector-valued special functions starting from spherical functions is contained in [6], [30], see also further references in these works. This approach differs from our standpoint based on spectral problems with multiple spectra.

For the classical index hypergeometric transform (1.3) several proofs of the inversion formula are known, see [13], [23], [32], 7.8.8. It is interesting to find a shorter proof of Theorem 1.1.

2.1. Hypergeometric operator. Let $\alpha \geqslant 0$, $\beta \in \mathbb{R}$. We consider the differential operator

$$D := \Bigl(\frac{1}{4} + x^2\Bigr)\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \frac{(\alpha + i\beta)^2}{4(1/2 + ix)} + \frac{(\alpha - i\beta)^2}{4(1/2 - ix)} + \frac{1}{4}. \tag{2.1}$$

This operator is formally self-adjoint in $L^2(\mathbb{R})$. Its resolvent $(D - \lambda)^{-1}$ is explicitly evaluated below. It is well defined for $\operatorname{Im}\lambda \neq 0$. Hence the deficiency indices of D are 0; this implies the self-adjointness of the operator D. Our purpose is to construct the spectral decomposition of the operator D.

Let

$$r(x) := (\tfrac{1}{2} + ix)^{-(\alpha+i\beta)/2}(\tfrac{1}{2} - ix)^{-(\alpha-i\beta)/2}.$$

Evaluating directly the differential operator

$$Bf := r^{-1}D(rf),$$

we obtain

$$B = \Bigl(\frac{1}{4} + x^2\Bigr)\frac{d^2}{dx^2} + \bigl(\beta + x(2 - 2\alpha)\bigr)\frac{d}{dx} + \bigl(\alpha - \frac{1}{2}\bigr)^2$$

Passing to the complex variable

$$z = 1/2 + ix,$$

we obtain the operator

$$A := -z(1-z)\frac{d^2}{dz^2} - \bigl(1 - \alpha - i\beta - z(2 - 2\alpha)\bigr)\frac{d}{dz} - \bigl(\alpha - \frac{1}{2}\bigr)^2. \tag{2.2}$$

Hence the equation $Af = \mu^2 f$ transforms to the hypergeometric equation

$$\Bigl[z(1-z)\frac{d^2}{dz^2} + \bigl(c - (a+b+1)z\bigr)\frac{d}{dz} - ab\Bigr]\, f = 0 \tag{2.3}$$

with

$$a = \tfrac{1}{2} - \alpha + \mu, \qquad b = \tfrac{1}{2} - \alpha - \mu, \qquad c := 1 - \alpha - i\beta$$

Now, we can use the Kummer series for solutions of the equation $Df = \mu^2 f$. Below we use the standard notation $u_1,\dots, u_6$ of [7] for the Kummer solutions of the standard hypergeometric equation (2.3) and use explicit formulae [7], (2.9.1)–(2.9.24) for their expansions in series.

2.2. Bases in the space of solutions of the hypergeometric equation. We will use four bases in the space of solutions of the equation $(D - \mu^2)f = 0$.

Basis S_1, S_2. Writing the Kummer solutions u_1, u_5 of the usual hypergeometric equation (2.3), we obtain the following pair of solutions S_1, S_2 of the equation (2.1):

$$S_1(\alpha,\beta;\mu;x) := \big(\tfrac12+ix\big)^{-(\alpha+i\beta)/2}\big(\tfrac12-ix\big)^{(\alpha+i\beta)/2-\mu-1/2}\times \\ \times\ {}_2F_1\left[\begin{matrix}1/2-\alpha+\mu,1/2-i\beta+\mu\\ 1-\alpha-i\beta\end{matrix};\frac{ix+1/2}{ix-1/2}\right]; \qquad (2.4)$$

$$S_2(\alpha,\beta;\mu;x) = S_1(-\alpha,-\beta;\mu;x).$$

The hypergeometric series in (2.4) is absolutely convergent for $\operatorname{Im} x > 0$, or equivalently, for $\operatorname{Re} z < 1/2$. On the line $x \in \mathbb{R}$ the series is conditionally convergent. Nevertheless it admits analytical continuation through the line $\operatorname{Im} x = 0$, and it is more convenient to think that we consider its analytic continuation.

Further, assume that for $x = iy$ with $-1/2 < y < 1/2$ (or, equivalently, for $0 < z < 1$)

$$\big(\tfrac12+ix\big)^{\lambda} := e^{\lambda\ln(1/2+ix)}, \qquad \big(\tfrac12-ix\big)^{\nu} := e^{\nu\ln(1/2-ix)}.$$

Now we can assume that our solutions S_1, S_2 are defined in the domain $\big\{\operatorname{Re} z \leqslant 1/2\big\} \setminus [-\infty, 0)$.

The solutions S_1, S_2 have asymptotics

$$S_1(x) \sim \big(\tfrac12+ix\big)^{(\alpha+i\beta)/2}, \qquad S_2(x) \sim \big(\tfrac12+ix\big)^{-(\alpha+i\beta)/2}; \qquad \text{for } x \to i/2. \qquad (2.5)$$

We also can define S_1 by the formula

$$\big(\tfrac12+ix\big)^{-(\alpha+i\beta)/2}\big(\tfrac12-ix\big)^{-(\alpha-i\beta)/2}\ {}_2F_1\left[\begin{matrix}1/2-\alpha+\mu,1/2-\alpha-\mu\\ 1-\alpha-i\beta\end{matrix};ix+1/2\right] \qquad (2.6)$$

Here the hypergeometric series converges for $|1/2-ix| < 1/2$, i.e., we do not obtain an explicit formula on the whole line $x \in \mathbb{R}$.

We mention that

$$S_1(\alpha,\beta;-\mu;x) = S_1(\alpha,\beta;\mu;x).$$

Basis T_1, T_2. Writing the pair of the Kummer solutions u_2, u_6, we obtain the following pair of solutions T_1, T_2 of the equation $(D-\mu^2)f = 0$

$$T_{1,2}(\alpha,\beta;\mu;x) := S_{1,2}(\alpha,-\beta;\mu;-x).$$

These solutions are defined in the domain $\big\{\operatorname{Re} z \geqslant 1/2\big\} \setminus [1,\infty)$.

Bases V_-, V_+ and W_-, W_+. The Kummer solutions u_3, u_4 are defined outside the circular lune $|z| < 1$, $|z-1| < 1$; in particular, this lune contains the segment $-\sqrt{3}/2 < x < \sqrt{3}/2$. Hence the analytic continuations of u_3, u_4 from the points $z = 1/2 + i\infty$ and $z = 1/2 - i\infty$ to the whole line $\operatorname{Re} z = 1/2$ are different. This gives the following two pairs $V_\pm$, $W_\pm$ of eigenfunctions of (2.1).

We define the solution V_- of the equation $(D-\mu^2)f=0$ as the solution that for $x>\sqrt{3}/2$ is given by the formula

$$V_-(\alpha,\beta;\mu;x)=e^{(-1/2+\alpha+i\beta-\mu)\pi i/2}\bigl(\tfrac12+ix\bigr)^{-(\alpha+i\beta)/2}\bigl(\tfrac12-ix\bigr)^{-1/2+(\alpha+i\beta)/2-\mu}\times \\ \times\ {}_2F_1\left[\begin{matrix}1/2-\alpha+\mu,1/2-i\beta+\mu\\ 1+2\mu\end{matrix};\frac{1}{1/2-ix}\right]. \quad (2.7)$$

Further, assume

$$V_+(\alpha,\beta;\mu;x)=V_-(\alpha,\beta;-\mu;x).$$

The asymptotics of $V_\pm$ for $x\to+\infty$ has the form

$$V_\pm\sim x^{1/2\pm\mu},\qquad x\to+\infty. \quad (2.8)$$

The solutions $W_\pm$ are determined by the condition

$$W_\pm\sim x^{1/2\pm\mu},\qquad x\to-\infty. \quad (2.9)$$

To obtain a formula for W_- for $x<\sqrt{3}/2$ we must change the sign in the argument of the exponential function in (2.7). Next, $W_+(\alpha,\beta;\mu;x)=W_-(\alpha,\beta;-\mu;x)$.

2.3. Transition matrices. Define the constants

$$C(\alpha,\beta;\mu):=\frac{\Gamma(\alpha+i\beta)\Gamma(1+2\mu)}{\Gamma(1/2+\alpha+\mu)\Gamma(1/2+i\beta+\mu)}; \quad (2.10)$$

$$\chi(\alpha,\beta;\mu):=e^{-(1/2+\alpha+i\beta-\mu)\pi i/2}. \quad (2.11)$$

In this notations (we use [7], (2.9.37), (2.9.39)),

$$V_-(\alpha,\beta;\mu;x)=C(\alpha,\beta;\mu)\chi(\alpha,\beta;\mu)S_1(\alpha,\beta;\mu;x)+ \\ +C(-\alpha,-\beta;\mu)\chi(-\alpha,-\beta;\mu)S_2(\alpha,\beta;\mu;x); \quad (2.12)$$

$$V_+(\alpha,\beta;\mu;x)=C(\alpha,\beta;-\mu)\chi(\alpha,\beta;-\mu)S_1(\alpha,\beta;\mu;x)+ \\ +C(-\alpha,-\beta;-\mu)\chi(-\alpha,-\beta;-\mu)S_2(\alpha,\beta;\mu;x). \quad (2.13)$$

Formulae expressing W_-, W_+ in S_1, S_2 can be obtain from (2.12)–(2.13) by the transform $\chi\mapsto\chi^{-1}$.

We also need in an expression of $W_\pm$ in terms of T_1, T_2. Similarly, applying [7], (2.9.38), (2.9.40), we obtain

$$W_-(\alpha,\beta;\mu;x)=-C(\alpha,-\beta;\mu)\chi^{-1}(-\alpha,\beta;-\mu)T_1(\alpha,\beta;\mu;x)- \\ -C(-\alpha,\beta;\mu)\chi^{-1}(\alpha,-\beta;-\mu)T_2(\alpha,\beta;\mu;x) \quad (2.14)$$

To obtain a formula for W_+ we must change sign of μ in $C(\alpha,\beta;\mu)$ and in $\chi(\alpha,\beta;\mu)$.

2.4. Evaluation of Wronskians. Denote by $\mathbf{wr}(P,Q):=\det\begin{pmatrix}P&Q\\P'&Q'\end{pmatrix}$ the Wronskian of two arbitrary solutions P, Q of the equation $(D-\mu^2)f=0$. This expression must have a form

$$\mathbf{wr}(P,Q)=\frac{\sigma(P,Q)}{1/4+x^2},$$

where $\sigma(P,Q)$ is a constant, see [10], I.17.1.

The value $\sigma(S_1,S_2)$ can be easily evaluated using asymptotics (2.5), we obtain

$$\sigma(S_1,S_2)=i(\alpha+i\beta).$$

Below, we need in $\sigma(V_-,W_-)$. The determinant Δ of the transition matrix from the basis (S_1,S_2) to the basis (V_-,W_-) can be easily evaluated, this gives

$$\sigma(V_-,W_-)=\Delta\cdot\sigma(S_1,S_2)=\frac{2\pi i C(\alpha,\beta;\mu)C(-\alpha,-\beta;\mu)}{\Gamma(\alpha+i\beta)\Gamma(-\alpha-i\beta)}. \tag{2.15}$$

2.5. Kernel of resolvent. Now we are ready to write the kernel $K(x,y;\lambda)$ of the resolvent

$$R(\lambda):=(D-\lambda)^{-1}$$

of the operator D,

$$R(\lambda)f(x)=\int_{-\infty}^{\infty}K(x,y;\lambda)f(y)\,dy.$$

Assume $\lambda\in\mathbb{C}\setminus(-\infty,0)$. Then the solution V_- is an element of $L^2(0,+\infty)$ and $W_-\in L^2(0,-\infty)$, see (2.8)–(2.9). Hence (see [5],XIII.3.6),

$$K(x,y;\lambda)=\begin{cases}\dfrac{V_-(x)W_-(y)}{\sigma(V_-,W_-)}, & \text{for } y<x\\[2mm] \dfrac{V_-(y)W_-(x)}{\sigma(V_-,W_-)}, & \text{for } x<y,\end{cases}$$

the value $\sigma(V_-,W_-)$ was evaluated above (2.15).

We intend to write an expansion of the differential operator R in eigenfunctions. According prescription rising to Weyl's work [33] (see [31] and detailed presentation in Dunford, Schwartz [5], XIII.5.18), we must evaluate the jump of the resolvent on the real line

$$\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\big(R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\big)\,d\lambda, \tag{2.16}$$

and represent it in the form

$$Lf(x)=\sum_{i=1,2}\sum_{j=1.2}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\sigma_i(x,\lambda)\overline{\sigma_j(y,\lambda))}\,f(y)\,d\mu_{ij}(\lambda)\Big)\,dy \tag{2.17}$$

for some solutions σ_1, σ_2 of the equation $Df=\lambda f$. Here $\mu_{ij}(\lambda)$ are (complex-valued) measures on $\mathbb{R}$. Then μ_{ij} is the spectral measure. Precisely, the operator

$$f\mapsto\Big(\int_{-\infty}^{\infty}f(x)\overline{\sigma_1(x)}\,dx,\int_{-\infty}^{\infty}f(x)\overline{\sigma_2(x)}\,dx\Big)$$

is a unitary operator from $L^2(\mathbb{R})$ to the space of $\mathbb{C}^2$-valued functions $(\varphi_1(\lambda), \varphi_2(\lambda))$ with the scalar product

$$\big\langle(\varphi_1(\lambda), \varphi_1(\lambda), (\psi_1(\lambda), \psi_1(\lambda)\big\rangle = \sum_{i=1,2} \sum_{i=1,2} \int_{-\infty}^{\infty} \varphi_i(\lambda)\overline{\varphi_j(\lambda)}\, d\mu_{ij}(\lambda)$$

(in particular, the right-hand side of this equality is a positive definite scalar product).

2.6. Formula for resolvent. To evaluate the jump of the resolvent for $\lambda \geqslant 0$ and for $\lambda \leqslant 0$, we are need of two explicit expressions for the kernel $K(x, y; \lambda)$ of the resolvent.

Expressing V_- via S_1, S_2 and W_- via T_1, T_2, we obtain the following expression for the resolvent

$$K(x, y; \lambda) = \frac{-1}{2\pi}\Gamma(\alpha{+}i\beta)\Gamma(-\alpha{-}i\beta)\Bigg[\frac{C(\alpha, -\beta; \sqrt{\lambda})}{C(-\alpha, -\beta; \sqrt{\lambda})} e^{(-1/2+\alpha-\sqrt{\lambda})\pi i} S_1(x)T_1(y) +$$
$$+\frac{C(-\alpha, \beta; \sqrt{\lambda})}{C(-\alpha, -\beta; \sqrt{\lambda})} e^{(-1/2+i\beta-\sqrt{\lambda})\pi i} S_1(x)T_2(y) + \frac{C(\alpha, -\beta; \sqrt{\lambda})}{C(\alpha, \beta; \sqrt{\lambda})} e^{(-1/2-i\beta-\sqrt{\lambda})\pi i} S_2(x)T_1(y)+$$
$$+ \frac{C(-\alpha, \beta; \sqrt{\lambda})}{C(\alpha, \beta; \sqrt{\lambda})} e^{(-1/2-\alpha-\sqrt{\lambda})\pi i} S_2(x)T_2(y)\Bigg], \quad (2.18)$$

where $C(\dots)$ is defined by (2.10).

Expressing V_-, W_- by S_1, S_2, we obtain

$$K(x, y; \lambda) = \frac{\Gamma(\alpha + i\beta)\Gamma(-\alpha - i\beta)}{2\pi C(\alpha, \beta; \sqrt{\lambda})\, C(-\alpha, -\beta; \sqrt{\lambda})} \times$$
$$\times \Big[C(\alpha, \beta; \sqrt{\lambda}))\, \chi(\alpha, \beta; \sqrt{\lambda})\, S_1(x) + C(-\alpha, -\beta; \sqrt{\lambda}))\, \chi(-\alpha, -\beta; \sqrt{\lambda}))\, S_2(x)\Big] \times$$
$$\times \Big[C(\alpha, \beta; \sqrt{\lambda}))\, \chi^{-1}(\alpha, \beta; \sqrt{\lambda})\, S_1(x) + C(-\alpha, -\beta; \sqrt{\lambda}))\, \chi^{-1}(-\alpha, -\beta; \sqrt{\lambda}))\, S_2(x)\Big], \quad (2.19)$$

where χ is given by (2.11).

These two expressions are meromorphic in the plane $\lambda \in \mathbb{C}$, with a cut on the negative semiaxis $\lambda < 0$.

2.7. Jump of resolvent for $\lambda > 0$. Evaluation of

$$L^{[0,\infty)} := \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_0^{\infty} \big(R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)\big)\, d\lambda \qquad (2.20)$$

is reduced to an evaluation of the residues. Expression (2.19) has simple poles at the points $\lambda = (\alpha - k - 1/2)^2$ for integer k, satisfying the condition $0 \leqslant k \leqslant \alpha - 1/2$. In addition, only the coefficient at $S_1(x)S_1(y)$ has nonzero residues, three other coefficients are holomorphic at these points.

Thus the integral operator (2.20) equals

$$L^{[0,\infty)}f(x)=\frac{1}{2\pi}\Gamma(\alpha+i\beta)\Gamma(\alpha-i\beta)\sum_{0\leqslant k<\alpha-1/2}\frac{(2\alpha-2k-1)\,(1-\alpha-i\beta)_k}{\Gamma(2\alpha-k)\,(1-\alpha+i\beta)_k\,k!}\times$$
$$\times\int_{-\infty}^{\infty}S_1(\alpha,\beta;\alpha-k-1/2;x)S_1(\alpha,\beta;\alpha-k-1/2;y)f(y)\,dy$$

Writing S_1 at the form (2.6), we obtain its representation in the terms of the Romanovski polynomials. Applying identity [7], (10.8.16) for Jacobi polynomials, we convert our expression to the form

$$\frac{1}{2\pi}\Gamma(\alpha+i\beta)\Gamma(\alpha-i\beta)\sum_{0\leqslant k<\alpha-1/2}\frac{2\alpha-2k-1}{\Gamma(2\alpha-k)\,k!}S_1(\alpha,\beta;\alpha-k-1/2;x)\times$$
$$\times\int_{-\infty}^{\infty}\overline{S_1(\alpha,\beta;\alpha-k-1/2;y)}f(y)\,dy$$

and this gives the required expression.

2.8. Jump of resolvent for $\lambda<0$. We must evaluate

$$K(x,y;\sqrt{\lambda})-K(x,y;-\sqrt{\lambda}),$$

assuming that λ is negative real and $\operatorname{Im}\sqrt{\lambda}>0$.

For definiteness, we present the calculation of the coefficient at $S_1(x)T_1(y)$. In three remaining cases, the evaluation is almost identical[7].

Thus, (see (2.18)), we intend to find

$$\frac{1}{-2\pi}\Gamma(\alpha+i\beta)\Gamma(-\alpha-i\beta)\times$$
$$\times\Bigl\{\frac{C(\alpha,-\beta;\sqrt{\lambda})}{C(-\alpha,-\beta;\sqrt{\lambda})}e^{-(1/2-\alpha+\sqrt{\lambda})\pi i}-\frac{C(\alpha,-\beta;-\sqrt{\lambda})}{C(-\alpha,-\beta;-\sqrt{\lambda})}e^{-(1/2-\alpha-\sqrt{\lambda})\pi i}\Bigr\}.$$

After direct cancellations, we obtain

$$\frac{1}{-2\pi}\Gamma(\alpha+i\beta)\Gamma(\alpha-i\beta)\times$$
$$\times\Bigl\{\frac{\Gamma(1/2-\alpha+\sqrt{\lambda})}{\Gamma(1/2+\alpha+\sqrt{\lambda})}e^{-(1/2-\alpha+\sqrt{\lambda})\pi i}+\frac{\Gamma(1/2-\alpha-\sqrt{\lambda})}{\Gamma(1/2+\alpha-\sqrt{\lambda})}e^{-(1/2-\alpha-\sqrt{\lambda})\pi i}\Bigr\}.$$

We convert this expression to

$$\frac{1}{-2\pi}\Gamma(\alpha+i\beta)\Gamma(\alpha-i\beta)\Gamma(1/2-\alpha+\sqrt{\lambda})\Gamma(1/2-\alpha-\sqrt{\lambda})\times$$
$$\times\Bigl\{\frac{e^{-(1/2-\alpha+\sqrt{\lambda})\pi i}}{\Gamma(1/2+\alpha+\sqrt{\lambda})\Gamma(1/2-\alpha-\sqrt{\lambda})}-\frac{e^{-(1/2-\alpha-\sqrt{\lambda})\pi i}}{\Gamma(1/2+\alpha-\sqrt{\lambda})\Gamma(1/2-\alpha+\sqrt{\lambda})}\Bigr\}.$$

[7] This is explained by natural symmetries $(\alpha,\beta)\sim(-\alpha,-\beta)\sim(i\beta,-i\alpha)$ of the equation (2.1).

Now, we will transform only the expression in the curly brackets

$$\pi\Big\{\dots\Big\} = \cos\big(\pi(\alpha+\sqrt{\lambda})\big)e^{-(1/2-\alpha+\sqrt{\lambda})\pi i} - \cos\big(\pi(\alpha-\sqrt{\lambda})\big)e^{-(1/2-\alpha-\sqrt{\lambda})\pi i}.$$

Next, we apply the Euler formulae for cos, collect similar terms in the linear combination of exponential functions, and again apply the Euler formulae. We obtain

$$i\sin(2\sqrt{\lambda}\pi i),$$

and this finishes the evaluation of the coefficient at $S_1(x)T_1(y)$.

After this, for the jump of the resolvent $L^{(-\infty,0]}$ on the semiaxis $(-\infty, 0]$, we obtain the expression of the form

$$L^{(-\infty,0)}f(x) = \sum_{i=1,2}\sum_{j=1,2}\int_{-\infty}^{0} d\lambda\Big\{\theta_{ij}(\lambda)\int_{-\infty} S_i(x)T_j(y)f(y)dy\Big\}$$

where θ_{ij} are some explicit products of Γ-functions.

Further, observe that for $\lambda < 0$ and $y \in \mathbb{R}$ we have $T_j(y) = \overline{S_j(y)}$. We obtain an expression of the form (2.17), and this finishes the evaluation of the spectral decomposition of the operator (2.1).

The parameter s from 1.4 is $\sqrt{\lambda}$.

§ 3. Construction of bases

3.1. One basis in $L^2(\mathbb{R})$. Fix real parameters p and q.

Lemma 3.1. *The system of functions*

$$r_{p,q}^{(n)}(x) := \big(\tfrac12 + ix\big)^{-1/2-n-(p+iq)/2}\big(\tfrac12 - ix\big)^{-1/2+n+(p-iq)/2}, \qquad (3.1)$$

where n ranges in $\mathbb{Z}$, is an orthonormal basis of $L^2(\mathbb{R}, dx/2\pi)$.

Proof. Pass to a new variable $\psi \in [0, 2\pi]$ by the formula

$$e^{i\psi} = \frac{1/2+ix}{1/2-ix}, \qquad d\psi = \frac{dx}{1/4+x^2}.$$

Our system of functions transforms to

$$e^{-in\psi}\cdot e^{-ip\psi/2}(2\cos\psi/2)^{iq}.$$

We obtain the standard orthogonal system $e^{-in\psi}$ up to multiplication by a function whose absolute value is 1. □

Emphasis also that orthogonality of our system can be obtained directly (since the scalar products can be easily evaluated using residue calculus).

3.2. Construction of bases. The bases (1.13) can be easily obtained, if we apply the double index hypergeometric transform to the orthogonal system (3.1).

Let us explain, how to perform the calculation. We must find

$$\int_{-\infty}^{\infty} r_{p,q}^{(n)}(x)\overline{Q_1(\alpha,\beta;x;s)}\,dx \tag{3.2}$$

where the function Q_1 is defined by the formula (1.6). For this purpose, we expand ${}_2F_1$ from formula (1.6) in the hypergeometric series in powers of

$$\frac{ix+1/2}{ix-1/2}$$

Then we integrate it termwise using the Cauchy beta-integral (see [26], v.1, (2.2.6.31) or [1], Chapter 1, ex. 13)

$$\int_{-\infty}^{\infty} \frac{dx}{(1/2+ix)^{\sigma}(1/2-ix)^{\tau}} = \frac{2\pi\Gamma(\sigma+\tau-1)}{\Gamma(\sigma)\Gamma(\tau)} \tag{3.3}$$

As a result, we obtain the following expression for (3.3)

$$\frac{2\pi\Gamma(1/2+iq-is)}{\Gamma(\alpha-i\beta)\Gamma((p+iq-\alpha+i\beta)/2+n+1-is)\Gamma((-p+iq+\alpha-i\beta+1)/2-n)}\times$$
$$\times\ {}_3F_2\left[{1/2-\alpha-is,1/2+i\beta-is,1/2+(p-iq-\alpha+i\beta)/2+n \atop 1-\alpha+i\beta,(p+iq-\alpha+i\beta)/2+n+1-is};1\right] \tag{3.4}$$

This expression is a variant of the final answer, its more symmetric form (1.13) can be obtained by Thomae transformation see [1], Corollary 3.3.6, [26], v.3, (7.4.4.2).

$${}_3F_2\left[{a,b,c \atop d,e};1\right] = \frac{\Gamma(d)\Gamma(e)\Gamma(r)}{\Gamma(a)\Gamma(b+r)\Gamma(c+r)}\ {}_3F_2\left[{d-a,e-a,r \atop b+r,c+r};1\right],$$

where $r=d+e-a-b-c$.

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Addendum. Perturbations of some classical hypergeometric orthogonal systems

Here we discuss orthogonal systems that can be obtained from some classical orthogonal system T_n by transformation $n \mapsto n + \theta$, there $\theta \in \mathbb{C}$ is fixed.

The most of facts formulated in this addendum are known or semi-known.

A.1. Spectral problems for one-parametric subgroups in $\mathrm{SL}(2, \mathbb{R})$**.** Consider a unitary irreducible representation of the universal covering of the group $\mathrm{SL}(2, \mathbb{R})$. There are three (up to a conjugation) one-parameter subgroups in $\mathrm{SL}(2, \mathbb{R})$, namely

– an elliptic subgroup $K = \mathrm{SO}(2)$, consisting of orthogonal matrices;

– a hyperbolic subgroup D consisting of diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, where $a > 0$

– a parabolic subgroup N consisting of matrices $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

Consider the spectral decomposition of our representation with respect to one of these subgroups (i.e., consider a realization of our representation in some space L^2, where our subgroup acts by multiplication by functions). Operators of representation of $\mathrm{SL}(2, \mathbb{R})$ in these models were obtained in the books of Vilenkin [Vil], Chapter 7, and of Vilenkin and Klimyk [VK1], Sections 6.8, 7.2, 7.6[8] [9]

Also consider some subgroup in $\mathrm{SL}(2, \mathbb{R})$ conjugated to $\mathrm{SO}(2)$, consider its eigenbasis in the space of representation, and consider the image of this eigenbasis in our space L^2. Obviously, we will obtain an orthogonal basis in the space L^2. Considering the spectral decomposition of a highest weight representation with respect to K, D or N, we obtain respectively Meixner polynomials, Meixner-Pollachek polynomials and Laguerre polynomials (see [VK1], 6.8.3, 7.7.11, 7.7.8).

Starting from a representation of a principal or complementary series, we obtain some nonpolynomial bases. These cases are discussed below in A.2-A.4.

Remark. In A.2 – A.4, we omit representations from our discussion and formally use only elementary analytic tools. But the representation theory allows to propose simple and effective actions. In fact, initial orthogonal bases, that we use below, are eigenbases for elliptic subgroups (for representations realized in functions on line or circle, see, for instance, [VK1]), the Fourier transform is the spectral expansion with respect to the parabolic subgroup N, the Mellin transform is the spectral expansion with respect to hyperbolic subgroup D, and expansion in Laurent series corresponds to expansion in eigenvectors of K.

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[8] To be precise, they usually consider the group $\mathrm{SL}(2, \mathbb{R})$ itself, extension of their formulae to the universal covering group is not a difficult problem.

[9] Likely, the earliest references that can be attributed to this subject are [Kep], [M-L].

A.2. Perturbation of Laguerre polynomials. Recall that the Kummer's confluent hypergeometric function ${}_1F_1$ is given by

$$ {}_1F_1[a;c;x] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n; $$

The confluent hypergeometric function of Tricomi (for details, see [HTF1],6.5.7 or [Sla1]) is given

$$ \Psi(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} {}_1F_1[a;c;x] + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} {}_1F_1[a-c+1; 2-c; x] $$

Fix $\alpha \in \mathbb{C}$. Consider the orthogonal basis

$$ \varphi_n(\alpha; x) = \bigl(1/2 + ix\bigr)^{-n-1/2-\alpha} \bigl(1/2 - ix\bigr)^{n-1/2+\overline{\alpha}} $$

in $L^2(\mathbb{R})$, where n ranges in $\mathbb{Z}$.

Considering the Fourier transforms of these functions, we obtain (see [IT1], III.3.2.12; see similar integrals in [Sla1]) the following proposition.

Proposition A.1. *The functions $\mathfrak{L}_n^{\alpha,\overline{\alpha}}(y)$ on $\mathbb{R}$ given by*

$$ \mathfrak{L}_n^{\alpha,\overline{\alpha}}(y) = \begin{cases} \dfrac{1}{\Gamma(-n+1/2-\overline{\alpha})} \Psi(1/2+n+\overline{\alpha}, 1+\overline{\alpha}-\alpha; y), & \textit{for } y>0 \\ \dfrac{1}{\Gamma(n+1/2+\alpha)} \Psi(1/2-n-\alpha, 1+\overline{\alpha}-\alpha; -y), & \textit{for } y<0 \end{cases} $$

form an orthonormal basis in L^2 on $\mathbb{R}$ with respect to the weight $\exp(-|y|)$.

Now, consider α, $\beta \in \mathbb{R}$, let

$$ |\alpha - \beta| < 1/2, \qquad \text{and } \alpha, \beta \neq \pm 1/2. $$

Define a weight $\mu_{\alpha,\beta}(y)\,dy$ on the line $\mathbb{R}$ by the formula

$$ \mu_{\alpha,\beta}(y) = |y|^{\beta-\alpha} e^{-y} \times \begin{cases} \cos(\pi\beta), & \text{for } y>0 \\ \cos(\pi\alpha), & \text{for } y<0 \end{cases}. $$

Define functions $\mathfrak{L}_n^{\alpha,\beta}$ on $\mathbb{R}$ by

$$ \mathfrak{L}_n^{\alpha,\beta}(y) = \begin{cases} \dfrac{1}{\Gamma(1/2-n-\alpha)} \Psi(1/2+n+\beta, 1-\alpha+\beta; y), & \text{for } y>0 \\ \dfrac{1}{\Gamma(1/2+n+\beta)} \Psi(1/2-n-\alpha, 1-\alpha+\beta; y), & \text{for } y<0 \end{cases}. $$

Proposition A.2. *The functions $\mathfrak{L}_n^{\alpha,\beta}$, where n range in $\mathbb{Z}$, form an orthogonal basis in the space $L^2(\mathbb{R}, \mu(y)\,dy)$. Moreover*

$$ \int_{-\infty}^{\infty} \bigl|\mathfrak{L}_n^{\alpha,\beta}(y)\bigr|^2 \mu(y)\,dy = \frac{(-1)^n \pi}{\Gamma(1/2+n+\beta)\Gamma(1/2-n-\alpha)} $$

To prove orthogonality, we write explicitly the orthogonality relations for $\mathcal{L}_n^{\alpha,\overline{\alpha}}(y)$ and apply the Kummer formula

$$\Psi(a,c;x) = x^{1-c}\Psi(a-c+1,2-c;x)$$

After this, we write the analytic continuation of orthogonality relations with respect to α, $\overline{\alpha}$.

Proposition A.1 corresponds to the principal series of representations of $\mathrm{SL}(2,\mathbb{R})$ and Proposition A.2 corresponds to the complementary series. The cases $\alpha = 1/2$ and $\beta = 1/2$ correspond to highest weight and lowest weight representations. In these cases, the system $\mathcal{L}_n^{\alpha,\beta}$ degenerates to the usual Laguerre (Sonin) system on half-line.

On these functions $\mathcal{L}_n$, see Groenevelt [Gro].

A.3. Perturbation of Meixner polynomials. We use the following nonstandard notation for the Gauss hypergeometric function.

$${}_2F_1^*[a,b;c;z] := \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\ {}_2F_1[a,b;c;z] = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\,k!} z^k$$

If $c = -l = 0, -1, -2, \dots$, when ${}_2F_1[a,b;c;z]$ has poles, but

$$\begin{aligned}{}_2F_1^*[a,b;c;z] &= \sum_{m=0}^{\infty} \frac{\Gamma(a+l+m+1)\Gamma(b+l+m+1)}{m!\,(m+l)!} z^k = \\ &= \frac{\Gamma(a+l+1)\Gamma(b+l+1)}{l!}\ {}_2F_1[a+l+1,b+l+1;l+1;z]\end{aligned}$$

(first $(l+1)$ summands in the series for ${}_2F_1$ are zero).

Consider the following orthogonal system

$$h_n(z) := z^n(\operatorname{ch} t - z\operatorname{sh} t)^{-1/2-\alpha-n}(\operatorname{ch} t - z^{-1}\operatorname{sh} t)^{-1/2+\alpha+n}$$

in L^2 on the circle $|z| = 1$; here n ranges in $\mathbb{Z}$. Expanding these functions in Fourier (Laurent) series, we obtain the following orthogonality relations for their Fourier coefficients.

Proposition A.3. *For fixed $\alpha \in \mathbb{C}$, $t \in \mathbb{R}$, the functions*

$$\mathcal{M}_n^{\alpha,\overline{\alpha}}(j) = \frac{(\operatorname{th} t)^{j-n}}{\Gamma(1/2-\overline{\alpha}-n)}\ {}_2F_1^*\begin{bmatrix}1/2+\alpha+j, 1/2-\overline{\alpha}-n \\ j-n+1\end{bmatrix}; \operatorname{th}^2 t\Big]$$

where n ranges in $\mathbb{Z}$, form an orthogonal basis in $l^2(\mathbb{Z})$. The inner squares of all the elements of the basis are $\operatorname{ch}^{-2} t$.

Proposition A.4. *Fix $t \in \mathbb{R}$, fix α, $\beta \in \mathbb{R}$ such that*

$$|\alpha| < 1/2, \quad |\beta| < 1/2$$

Consider the space $H_{\alpha,\beta}$ of two-side sequences $x(j)$ with the scalar product

$$\langle x, y\rangle = \sum_{j=-\infty}^{\infty} x(j)\overline{y(j)} \cdot \frac{\Gamma(1/2+\alpha+j)}{\Gamma(1/2+\beta+j)}$$

Then the system of functions

$$\mathcal{M}_n^{\alpha,\beta}(j) := \frac{(\operatorname{th} t)^{j-n}}{\Gamma(1/2-\beta-n)}\ {}_2F_1^*\left[\begin{matrix}1/2+\alpha+j, 1/2-\alpha-n\\ j-n-1\end{matrix}; \operatorname{th}^2 t\right]$$

forms an orthogonal basis in $H_{\alpha,\beta}$*, and*

$$\sum_{-\infty}^{\infty}\left|\mathcal{M}_n^{\alpha,\beta}(j)\right|^2\cdot\frac{\Gamma(1/2+\alpha+j)}{\Gamma(1/2+\beta+j)} = \operatorname{ch}^{-2} t\,\Gamma^2(1/2-\alpha-n)$$

Proof. Consider the following biorthogonal system on the circle

$$u_n(z) = z^n(\operatorname{ch} t - z\operatorname{sh} t)^{-1/2-\alpha-n}(\operatorname{ch} t - z^{-1}\operatorname{sh} t)^{-1/2+\beta+n};$$
$$v_n(z) = z^n(\operatorname{ch} t - z\operatorname{sh} t)^{-1/2-\overline{\beta}-n}(\operatorname{ch} t - z^{-1}\operatorname{sh} t)^{-1/2+\overline{\alpha}+n}$$

Expanding these function in Fourier (or Laurent) series, we obtain a biorthogonal system of two-side sequences. Images $\widehat{u}_n$, $\widehat{v}_n$ of u_n, v_n seem different at the first glance. But applying the identity (see [HTF1], (2.1.23))

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a, c-b; c; z)$$

we observe that $\widehat{u}_n$, $\widehat{v}_n$ coincide modulo a simple elementary factor. Thus we obtain orthogonality relations for u_n. □

These facts are contained in Vilenkin–Klimyk [VK1] (they discuss only two parametric family of bases), apparently, these bases firstly appeared in [VK0].

A.4. Perturbation of the Meixner–Pollachek system. Consider the Mellin transform in $L^2(\mathbb{R})$, i.e., for $g \in L^2(\mathbb{R})$, we define two functions

$$\widehat{g}_1(s) := \int_0^{\infty} x^{is-1/2} g(x)\,dx$$
$$\widehat{g}_2(s) := \int_{-\infty}^{0} (-x)^{is-1/2} g(x)\,dx$$

on $\mathbb{R}$. We have the following condition of unitarity

$$\int_{-\infty}^{\infty}\left|g(x)\right|^2 dx = \frac{1}{2\pi}\Big\{\int_{-\infty}^{\infty}\left|\widehat{g}_1(s)\right|^2 dx + \int_{-\infty}^{\infty}\left|\widehat{g}_2(s)\right|^2 dx\Big\}$$

Fix $\alpha = \sigma + i\tau \in \mathbb{C}$ and $\varphi \in (0,\pi)$. Consider the orthogonal basis in $L^2(\mathbb{R})$ given by

$$h_n(x) = (1+xe^{i\varphi})^{-1/2-\sigma-i\tau-n}(1+xe^{-i\varphi})^{-1/2+\sigma-i\tau+n}x^{i\tau}$$

We evaluate Fourier transform of these functions (see [PBM1], 2.2.6.24) and obtain the following construction.

Consider the Hilbert space V_α, whose elements are pairs of functions (f_1, f_2), depending in the variable $s \in \mathbb{R}$, and the inner product is

$$\bigl\langle (f_1, f_2),\, (g_1, g_2) \bigr\rangle = \frac{|\Gamma(1/2 + i\tau + is)\Gamma(1/2 + i\tau - is)|^2}{2\sin\varphi |\Gamma(1 + 2i\tau)|^2} e^{(s+\tau)\varphi} \times \\ \times \Bigl\{ \int_{-\infty}^{\infty} f_1(s)\overline{g_1(s)}\, ds + e^{-\pi s} \int_{-\infty}^{\infty} f_2(s)\overline{g_2(s)}\, ds \Bigr\}$$

Consider the hypergeometric series

$${}_2F_1\left[\begin{matrix} 1/2 + is + i\tau, 1/2 - \sigma + i\tau - n \\ 1 + 2i\tau \end{matrix}; u\right]$$

We consider values of analytic continuation of this series at the point

$$u = 1 - e^{-2i\varphi}, \qquad \text{where } 0 < \varphi < \pi$$

By $G_n^1(s, \varphi)$ we denote the value obtained after passing the way $h(\theta) := 1 - e^{-2i\theta}$, where $\theta \in [0, \varphi]$. By $G_n^2(s; \varphi)$ we obtain the result of passing $h(\theta) := 1 - e^{2i\theta}$, where $\theta \in [0, \pi - \varphi]$.

PROPOSITION A.5 *The system $(G_n^1(s), G_n^2(s))$ is an orthonormal basis of the space V_α.*

Thus, we obtain a $\mathbb{C}^2$-valued orthogonal system (as above in Section 3).

A.5. On properties of perturbed systems. Standard properties of Laguerre, Meixner-Pollachek, and Meixner polynomials (see [HTF2] or [KS]) can be easily extracted from the theory of highest weight representations of $\mathrm{SL}(2, \mathbb{R})$; this is explained in Vilenkin–Klimyk [VK1]. It is clear that their methods remain valid for the perturbed systems.

One of nice facts that can be obtained in this way is surviving of the Meixner generating function (see [Mei]). For instance, consider the Poisson kernel for perturbed Meixner–Pollachek system,

$$\mathcal{K}_{\alpha,\varphi}(s, t; \theta) = \sum_{n=-\infty}^{\infty} \begin{pmatrix} G_n^1(s)\,\overline{G_n^1(t)} & G_n^1(s)\,\overline{G_n^2(t)} \\ G_n^2(s)\,\overline{G_n^1(t)} & G_n^2(s)\,\overline{G_n^2(t)} \end{pmatrix} e^{in\theta}$$

This sum admits an explicit evaluation, since $\mathcal{K}(s, t; \theta)$ are the kernels of integral operators corresponding to matrices

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

in the D-diagonal model of the representation. The latter kernels (for $2\alpha \in \mathbb{Z}$) were evaluated by Vilenkin [Vil], (7.4.1.13)–(7.4.1.16) (see also [VK1], (7.2.1.12)–(7.2.1.15)).

The same is valid for the perturbed Meixner system and perturbed Laguerre system (for the Laguerre system, this gives an extension of Myller-Lebedeff bilinear generating function, see [HTF2], [M-L]).

A.5. Perturbed Jacobi systems and Molchanov's singular boundary problems. Let $-1 < \alpha < 1$, $\beta > -1$. Fix θ, such that $0 < \theta < 1$. We also assume $0 < \theta + \alpha < 1$.

Let $n - \theta$ be integer. We consider a function Φ_n on the half-line $x > 0$ given by

$$\Phi_n(x) = \begin{cases} {}_2F_1\left[\begin{matrix}-n, n+\alpha+\beta+1\\ \beta+1\end{matrix}; x\right], & \text{for } 0 < x < 1\\ x^{-n-\beta-1}(x-1)^{-\alpha}\Gamma\left[\begin{matrix}n+\alpha+\beta+1,\ \alpha+n+1\\ 2n+\alpha+\beta+2\end{matrix}\right]\times & \\ \qquad\times\ {}_2F_1\left[\begin{matrix}n+\alpha+\beta+1, 2n+\alpha+1\\ 2n+\alpha+\beta+2\end{matrix}; \frac{1}{x}\right], & \text{for } x > 1\end{cases}$$

Remark. These two functions are Kummer solutions [HTF1], (2.9.11), (2.9.14) of the same differential hypergeometric equation.

Theorem A.6. *The system Φ_n, where $n - \theta \in \mathbb{Z}$ and $2n + \alpha + \beta + 1 > 0$, is orthogonal with respect to the weight*

$$w(x) = \begin{cases} x^\beta (1-x)^\alpha, & \text{for } 0 < x < 1;\\ \dfrac{\sin(\pi\theta)}{\sin\pi(\alpha+\theta)} x^\beta (x-1)^\alpha & \text{for } x > 1\end{cases}$$

and

$$\int_0^\infty |\Phi_n(x)|^2\, dx = \frac{\Gamma^2(\beta+1)\Gamma(1+n)\Gamma(1+n+\alpha)}{(\alpha+\beta+2n+1)\Gamma(\beta+1+n)\Gamma(n+\alpha+\beta+1)}$$

Remark. *We emphasis that the orthogonal system Φ_n is not a basis.*

Proof. Denote by $H(x)$ the Heaviside one-step function, $H(x) = 1$ for $x > 0$, and $H(x) = 0$ for $x < 0$. Consider two Mellin–Barnes integrals

$$K_1(x) = \frac{1}{2\pi i}\Gamma\left[\begin{matrix}\beta+1,\ 1+m+\alpha\\ \beta+m+1\end{matrix}\right]\int_{-i\infty}^{+i\infty}\Gamma\left[\begin{matrix}s,\ \beta+1+m-s\\ m+\alpha+1+s, \beta+1-s\end{matrix}\right] x^{-s} ds =$$
$$= {}_2F_1\left[\begin{matrix}\beta+1+m,\ -m-\alpha\\ \beta+1\end{matrix}; x\right] H(1-x) +$$
$$+ x^{-\beta-1-m}\Gamma\left[\begin{matrix}\beta+1,\ 1+m+\alpha\\ -m, 2m+\alpha+\beta+2\end{matrix}\right]\ {}_2F_1\left[\begin{matrix}\beta+1+m,\ m+1\\ 2m+\alpha+\beta+2\end{matrix}; \frac{1}{x}\right] H(x-1)$$

$$K_2(x) = \frac{1}{2\pi i}\Gamma\left[\begin{matrix}2n+\alpha+\beta+2,\ -\alpha-n\\ n+\alpha+\beta+1\end{matrix}\right]\int_{-i\infty}^{+i\infty}\Gamma\left[\begin{matrix}\alpha+n+s,\ \beta+1-s\\ s,\ n+\beta+2-s\end{matrix}\right] x^{-s} ds =$$
$$= x^{n+\alpha}\ {}_2F_1\left[\begin{matrix}n+\alpha+\beta+1,\ \alpha+n+1\\ 2n+\alpha+\beta+2\end{matrix}; x\right] H(1-x) +$$
$$+ x^{-1-\beta}\Gamma\left[\begin{matrix}2n+\alpha+\beta+2, -\alpha-n\\ n+1\end{matrix}\right]\ {}_2F_1\left[\begin{matrix}n+\alpha+\beta+1, n+1\\ \beta+1\end{matrix}; \frac{1}{x}\right]$$

These formulae can be easily checked using the standard technic of the Mellin–Barnes integrals, see [Sla2] (see also tables [PBM3], 8.4.49).

We intend to evaluate

$$\int_0^\infty K_1(x) K_2\Big(\frac{1}{x}\Big)\frac{dx}{x}$$

(this integral is the inner product of Φ_m and Φ_n up to a Γ-factor).

Since the Mellin transform transfer a convolution to a product, we must evaluate

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \Gamma\begin{bmatrix} s,\, \beta+1+m-s \\ m+\alpha+1+s, \beta+1-s\end{bmatrix} \Gamma\begin{bmatrix}\alpha+n+s,\, \beta+1-s \\ s,\, n+\beta+2-s\end{bmatrix} ds =$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \Gamma\begin{bmatrix} \beta+1+m-s,\, \alpha+n+s \\ m+\alpha+1+s, n+\beta+2-s\end{bmatrix} ds =$$

$$= \Gamma\begin{bmatrix} \alpha+\beta+n+m+1,\, 1 \\ \alpha+\beta+n+m+2,\, 1+m-n,\, 1+n-m\end{bmatrix}$$

(see [PBM3]). The final expression is zero if $m \ne n$ and $m-n\in\mathbb{Z}$.

Remark. Let us explain why our calculation is not artificial. Let us evaluate the inner product of two Jacobi polynomials

$$\int_0^1 {}_2F_1\Big[\begin{matrix}-n, n+\alpha+\beta+1 \\ \beta+1\end{matrix}; x\Big]\ {}_2F_1\Big[\begin{matrix}-m, m+\alpha+\beta+1 \\ \beta+1\end{matrix}; x\Big] x^\beta (1-x)^\alpha\, dx$$

in the following way. We represent the integral as a multiplicative convolution of

$$K_1(x) = (1-x)^\alpha\ {}_2F_1\Big[\begin{matrix}-m, m+\alpha+\beta+1 \\ \beta+1\end{matrix}; x\Big]\, H(1-x)$$

$$K_2(x) = x^{-\beta-1}\ {}_2F_1\Big[\begin{matrix}-n, n+\alpha+\beta+1 \\ \beta+1\end{matrix}; \frac{1}{x}\Big]\, H(x-1)$$

and evaluate the convolution using the Mellin–Barnes integral representations of Jacobi polynomials (see [PBM3], 8.4). The calculation obtained in this way admits analytic continuation to noninteger number n of "Jacobi polynomial" $P_n^{\alpha,\beta}$. This gives the required formula for K_1, K_2. $\square$

Orthogonal systems of this type appear in the following situation. Consider the Laplace operator Δ on a pseudo-Riemannian symmetric space G/H. Restricting Δ to the subspace of H-invariant functions, we obtain an ordinary second-order differential operator. This operator is the usual hypergeometric operator (2.3) on some contour passing through one or more singular points of the hypergeometric equation (different series of symmetric spaces give different contours). Molchanov obtained explicit spectral decomposition for several boundary problems of this type (see [Mol1], [Mol2]). These singular differential operators have countable discrete spectra and hence we obtain (noncomplete)

orthogonal systems consisting of piece-wise Gauss hypergeometric functions. Our construction is not covered by such examples and also does not cover them.

A.6. Perturbation of Wilson system. Koornwinder [Koo] observed that the Wilson ${}_4F_3$ orthogonal system can be obtained by application of the index hypergeometric transform to Jacobi polynomials. It is possible to apply this method to a perturbed Jacobi system. These gives (noncomplete) orthogonal systems, whose elements are pairs of ${}_4F_3$-functions.

A.7. An example of ${}_2F_2$-basis. Since we discuss possibilities of obtaining of hypergeometric bases using standard integral transforms, let us construct a basis that is not a deformation of a classical system.

Fix $\rho > -1$. Consider the function r_n on $\mathbb{R}$ given by

$$r_n = \begin{cases} (1-x)^{\rho/2} P_n^{\rho,0}(x) & \text{for } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where $P_n^{\rho,0}$ is a Jacobi polynomial. Obviously, the functions $r_n(x+2m)$, where $n = 0, 1, 2, \dots$ and $m \in \mathbb{Z}$, form an orthogonal basis in $L^2(\mathbb{R})$.

Evaluating their Fourier transforms (see [PBM1], 2.22.5.1), we obtain the following statement

PROPOSITION A.7 *The functions*

$$e^{-2mi\xi}\, {}_2F_2\left[\begin{matrix} \rho/2+1,\ 1-\rho/2 \\ \rho/2+n+2, -\rho/2+1-n \end{matrix}; 2i\xi\right]$$

where $n = 0, 1, 2, \dots$ and $m \in \mathbb{Z}$, form an orthonormal basis in $L^2(\mathbb{R})$.

Our expression ${}_2F_2$ is relatively simple. Up to a constant (depending on n) it equals

$$\sum_{k=0}^{\infty} \frac{(\rho/2+1+k)_n}{(-\rho/2+1-n+k)_n\, k!} (2i\xi)^k$$

REMARK. A toy construction in the same type can be obtained by applying the Fourier transform to the standard trigonometric system e^{inx} on $[-\pi, \pi]$. Thus the functions $\sin(y\pi)e^{2i\pi my}/(n+y)$, where $m \in \mathbb{Z}$, $n = 0, 1, 2, \dots$ form an orthonormal basis in $L^2(\mathbb{R})$.

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